A coarse Cartan-Hadamard theorem with application to the coarse Baum-Connes conjecture

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arXiv:1705.05588 (same title)
with OGUNI Shin-ichi （尾國新一）

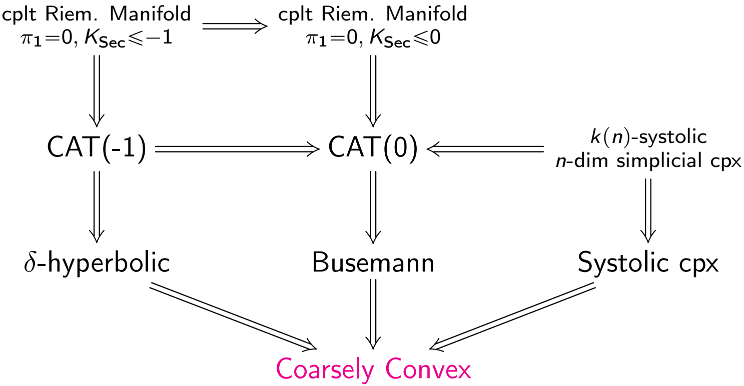
Table of contents

Several notions of non-positively/negatively curved spaces

Class	by	QI-inv	Product	coarse Baum-Connes
Geodesic δ-hyperbolic	Gromov	Yes	No	Higson-Roe, Willett
CAT(0)	C-A-T Gromov	No	Yes	Higson-Roe, Willett F-O
Busemann	Busemann	No	Yes	Higson-Roe, Willett F-O
Systolic complex	Chepoi J-S, H	No	No $\mathbb{R} \times \mathbb{R}^2$	Novikov: O-P cBC: F-O
Coarsely Convex	F-O	Yes	Yes	F-O

J-S: Januszkiewich-Świątkowski   H: Haglund
O-P: Osajda-Przytycki

Relations


cplt Riem. Manifold
$\pi_1=0$, $K_{\text{Sec}} \leqslant -1$
cplt Riem. Manifold
$\pi_1=0$, $K_{\text{Sec}} \leqslant 0$
CAT(-1)
CAT(0)
$k(n)$-systolic
n-dim simplicial cpx
δ-hyperbolic
Busemann
Systolic cpx
Coarsely Convex

Some notations

- Let (X, d) be a metric space.
- An isometry $\gamma\colon [a, b] \to X$ is called a geodesic segment.
- (X, d) is a geodesic space if any two points in X is connected by a geodesic segment.
- For $p, q \in X$, we denote by $\overline{p, q} := d(p, q)$ the distance between p and q.

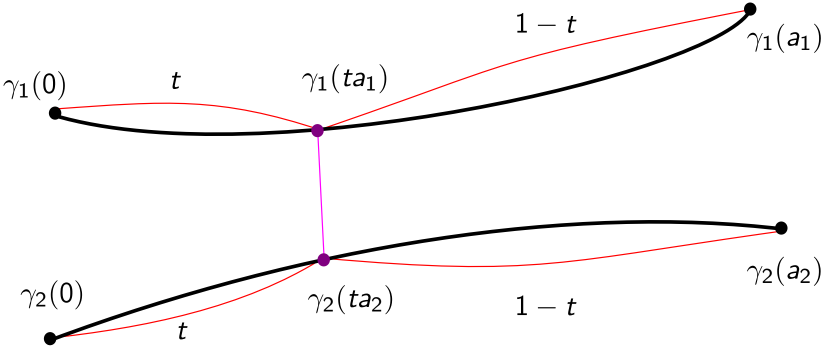
Convexity of Metric

Definition

The metric d of X is convex $\Leftrightarrow$
$\forall \gamma_i \colon [0, a_i] \to X$ geodesic segments ($i = 1, 2$), $\forall t \in [0, 1]$ we have

$$\overline{\gamma_1(ta_1), \gamma_2(ta_2)} \leqslant (1 - t)\, \overline{\gamma_1(0), \gamma_2(0)} + t\, \overline{\gamma_1(a_1), \gamma_2(a_2)} .$$

Remark: X is a Busemann space $\Leftrightarrow$ (X, d) is a geodesic space and d is convex.



QI-invariance

Clearly this property is NOT Quasi-Isometry-invariant.

We want to make it QI-invariant!

QI-invariance: Naive Idea

Naive Idea: Replace GEODESIC by (λ, k)-QUASI-GEODESIC and introduce some constants E,C.

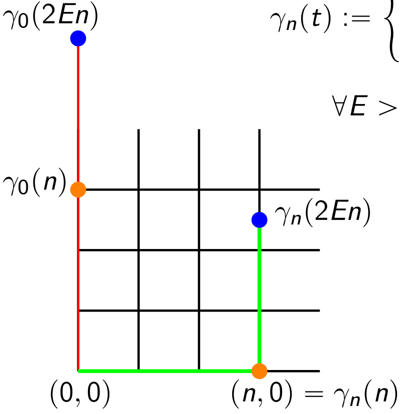
$\forall \gamma_i \colon [0, a_i] \to X$ (λ, k)-quasi-geodesic $(i = 1, 2)$, $\forall t \in [0, 1]$ we have

$$\overline{\gamma_1(ta_1), \gamma_2(ta_2)} \leqslant (1-t)E\,\overline{\gamma_1(0), \gamma_2(0)} + tE\,\overline{\gamma_1(a_1), \gamma_2(a_2)} + C.$$

$\cdots$ This does not work!

Example

- $\mathbb{R}^2 \overset{\text{QI}}{\cong} \text{Cay}(\mathbb{Z}^2, \{(1,0),(0,1)\}) \overset{\text{QI}}{\cong} (\mathbb{R}^2, l_1)$ (l_1: Manhattan metric)
- For $n \in \mathbb{Z}_{\geqslant 0}$, define $\gamma_n : \mathbb{R}_{\geqslant 0} \to R^2$ by

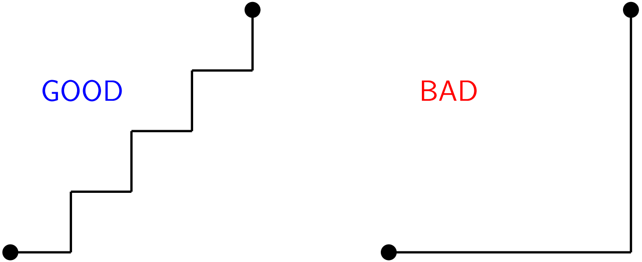


$$\gamma_n(t) := \begin{cases} (t,0) & \text{when}(t \leqslant n) \\ (n, t-n) & \text{when}(t > n) \end{cases}$$

$\forall E > 1$ fixed, we have

$$\overline{\gamma_0(n), \gamma_n(n)} - \frac{1}{2E} E\, \overline{\gamma_0(2En), \gamma_n(2En)} = 2n - n = n \to \infty$$

- This does not work because
  there exists MANY QUASI-GEODESICS.



- IDEA: Consider ONLY “GOOD” quasi-geodesics.

Theorem (Osajda-Przytycki)

Let X be a systolic complex.
Then X has a family of good geodesics.

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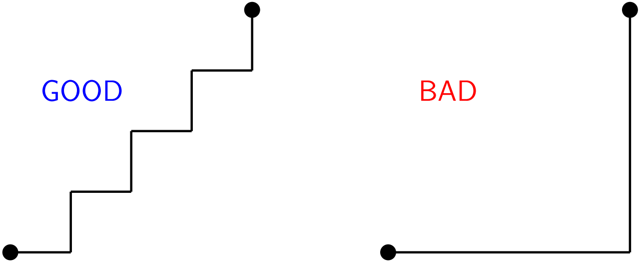

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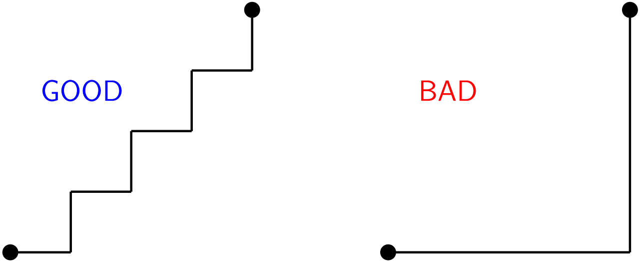

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Let X be a systolic complex.
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Coarsely Convex space

Definition

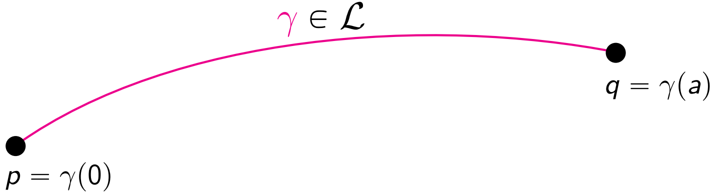
- Let X be a metric space.
- Let $\lambda \geqslant 1$, $k \geqslant 0$, $E \geqslant 1$, and $C \geqslant 0$ be constants.
- Let $\theta \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non-decreasing function.
- Let $\mathcal{L}$ be a family of (λ, k)-quasi-geodesic segments.

The metric space X is $(\lambda, k, E, C, \theta, \mathcal{L})$-*coarsely convex*, if $\mathcal{L}$ satisfies the three conditions in the following slides.

First: $\mathcal{L}$-Connected

- $\lambda \geqslant 1$, $k \geqslant 0$, $E \geqslant 1$, $C \geqslant 0$: constants.
- $\mathcal{L}$: a family of (λ, k)-quasi-geodesic segments.

(i) $\forall p, q \in X$, $\exists \gamma \in \mathcal{L}$ with $Domain(\gamma) = [0, a]$, s.t. $\gamma(0) = p$, $\gamma(a) = q$.



Second: Coarsely Convex Inequality

- $\lambda \geqslant 1$, $k \geqslant 0$, $E \geqslant 1$, $C \geqslant 0$: constants.
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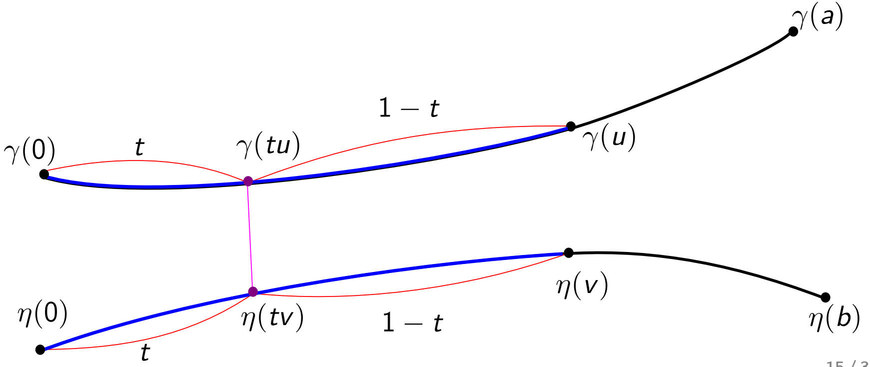
(ii) $\forall \gamma, \eta \in \mathcal{L}$ with $Domain(\gamma) = [0, a]$, $Domain(\eta) = [0, b]$.
For $u \in [0, a]$, $v \in [0, b]$, and $0 \leqslant t \leqslant 1$, we have

$$\overline{\gamma(tu), \eta(tv)} \leqslant (1-t)E\,\overline{\gamma(0), \eta(0)} + tE\,\overline{\gamma(u), \eta(v)} + C.$$



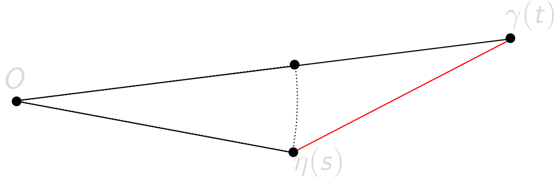
Third: Regularity of Parameters

- $\theta\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$: a non-decreasing function.
- $\mathcal{L}$: a family of (λ, k)-quasi-geodesic segments.

(iii) $\forall \gamma, \eta \in \mathcal{L}$ with $Domain(\gamma) = [0, a]$, $Domain(\eta) = [0, b]$.
For $t \in [0, a]$ and $s \in [0, b]$, we have

$$|t - s| \leqslant \theta(\,\overline{\gamma(0), \eta(0)} + \overline{\gamma(t), \eta(s)}\,).$$

Consider the case $\gamma(0) = \eta(0) = O$.



If γ, η are geodesic, then by triangle inequality,

$$|t - s| = |\,\overline{\gamma(0), \gamma(t)} - \overline{\eta(0), \eta(s)}\,| \leqslant \overline{\gamma(t), \eta(s)}$$

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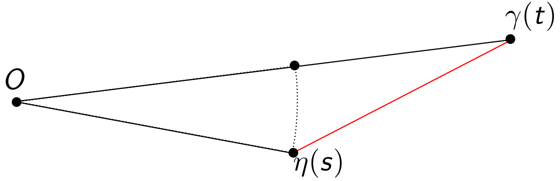

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Remark

If X is a

- *Gromov hyperbolic space,*
- *Busemann space, or*
- *Systolic complex,*

then we can take $\mathcal{L}$ a family of geodesic segments. Therefore the third condition is satisfied.

Coarsely Convex

In the above definition, the family $\mathcal{L}$ satisfying (i), (ii), and (iii) is called a system of good quasi-geodesic segments.

We say that a metric space X is coarsely convex if it is $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex for some $\lambda, k, E, C, \theta, \mathcal{L}$.

Basic properties

Proposition (QI-invariant)

- *Let X and Y be metric spaces.*
- *Suppose that X and Y are quasi-isometric.*

Then X is coarsely convex $\Leftrightarrow$ Y is coarsely convex.

Proposition (Stable under direct products)

- *Let X and Y be metric spaces.*
- *Suppose that X and Y are coarsely convex*

Then the direct product $X \times Y$ is coarsely convex.

Examples

The following metric spaces are coarsely convex.

- Geodesic Gromov hyperbolic spaces.
- CAT(0)-spaces.
- Busemann spaces.

Theorem (Osajda-Przytycki)

Systolic complexes are coarsely convex.

Theorem (Osajda-Huang, Osajda-Prytuła)

Artin groups of large type and graphical $C(6)$ small cancellation groups are systolic groups. i.e. Each of them acts geometrically on a systolic complex. Especially, they are coarsely convex.

Moreover, the direct products of the above spaces and groups are coarsely convex.

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Visual boundary

- Let X be a coarsely convex space with the system of good quasi-geodesic segments $\mathcal{L}$.
- We say that the map $\gamma\colon \mathbb{Z}_{\geqslant 0} \to X$ is $\mathcal{L}$-approximatable if $\exists\{\gamma_n\} \subset \mathcal{L}$ such that γ_n converges to γ uniformly on $\{0, 1, \ldots, l\}$ for all $l \in \mathbb{Z}_{\geqslant 0}$.
- We define

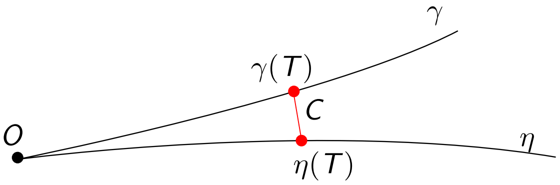
$$\partial X := \{\gamma\colon \mathbb{Z}_{\geqslant 0} \to X : \gamma \text{ is } \mathcal{L}\text{-approximatable}\}/\sim$$

where $\gamma \sim \eta$ if $\sup\{\overline{\gamma(t), \eta(t)} : t \in \mathbb{Z}_{\geqslant 0}\} < \infty$.

Gromov Product

- Choose a base point $O \in X$.
- For $\gamma, \eta \colon \mathbb{Z}_{\geqslant 0} \to X$: $\mathcal{L}$-approximatable, $\gamma(0) = O$, we define

$$(\gamma|\eta) := \sup\left\{ T : \overline{\gamma(T), \eta(T)} \leqslant C \right\}.$$



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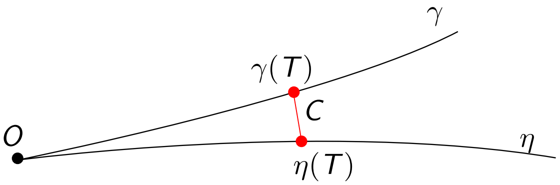

We recall that C appears in the coarsely convex inequality:

$$\overline{\gamma(tu), \eta(tv)} \leqslant (1-t)E\,\overline{\gamma(0), \eta(0)} + tE\,\overline{\gamma(u), \eta(v)} + C.$$

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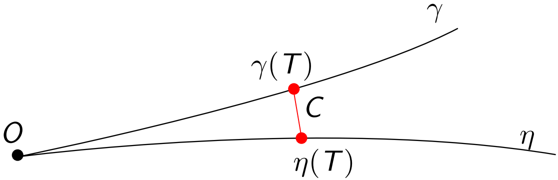

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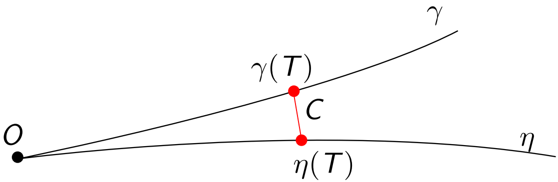

We define $\rho([\gamma],[\eta]) := \dfrac{1}{(\gamma|\eta)}$. This is NOT metric.

Lemma

$\exists D > 1$ s.t. for γ, η, ξ: $\mathcal{L}$-approximatable rays starting at O,

$$\rho([\gamma],[\xi]) \leqslant D \max\{\rho([\gamma],[\eta]), \rho([\eta],[\xi])\}$$

There is a standard recipe to deform ρ to a METRIC.

Proposition

$\exists d_{\partial X}$:metric on ∂X & $0 < \exists \epsilon \leqslant 1$ s.t. $\forall [\gamma], [\eta] \in \partial X = \mathcal{L}_O^\infty / \sim$,

$$\frac{1}{2D^\epsilon}\rho([\gamma]|[\eta])^\epsilon \leqslant d_{\partial X}([\gamma],[\eta]) \leqslant \rho([\gamma]|[\eta])^\epsilon$$

Proposition

X is proper $\Rightarrow \partial X$ is compact.

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X is proper $\Rightarrow \partial X$ is compact.

Coarse Cartan-Hadamard Theorem

Let X be a proper coarsely convex space. The open cone over ∂X is

$$\mathcal{O}\partial X := [0,\infty) \times \partial X / \{0\} \times \partial X$$

with metric: for $t, s \in [0,\infty)$; $x, y \in \partial X$

$$\overline{tx, sy} := |t - s| + \min\{t, s\} d_{\partial X}(x, y)$$

Theorem (coarse Cartan-Hadamard)

The "exponential" map

$$\exp\colon \mathcal{O}\partial X \ni t[\gamma] \to \gamma(r(t)^{\frac{1}{\epsilon}}) \in X$$

is coarsely homotopy equivalent map. Especially, $\mathcal{O}\partial X$ and X are coarsely homotopy equivalent.

Here $r\colon [0,\infty) \to [0,\infty)$ is a contraction such that $r(t) \to \infty$ as $t \to \infty$.

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the Coarse Baum-Connes conjecture

- Y : proper metric space
- $KX_\bullet(Y)$: coarse K-homology of Y

  (ex. $KX_\bullet(\mathbb{Z}^n) \cong KX_\bullet(\mathbb{R}^n) \cong K_\bullet(\mathbb{R}^n)$)
- $C^*(Y)$: a C^*-algebra constructed from Y, called Roe algebra, which is a non-equivariant analog of the reduced group C^*-algebra.

Conjecture (coarse Baum-Connes)

The following coarse assembly map is an isomorphism.

$$\mu_Y\colon KX_\bullet(Y) \to K_\bullet(C^*(Y)).$$

Corollary (of Main theorem)

Coarse Baum-Connes conjecture holds for proper coarsely convex spaces, especially, for locally finite systolic complexes.

Example

The above corollary covers following spaces and groups.

- Proper Geodesic Gromov hyperbolic spaces.
- Proper CAT(0)-spaces, more generally, Busemann spaces.
- Artin groups of large types (NEW!).
- graphical $C(6)$-small cancellation groups (NEW!).
- Direct product of above spaces and groups (NEW!).

Remark

Osajda-Przytycki showed that Novikov conjecture for systolic groups holds.

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Proof of Corollary: Coarse Homotopy Invariance

Since $\exp\colon X \to \mathcal{O}\partial X$ is a coarsely homotopy equivalent map, following diagram is commutative and two vertical arrows are isomorphisms.

$$\begin{array}{ccc} KX_\bullet(\mathcal{O}\partial X) & \xrightarrow{\mu_{\mathcal{O}\partial X}} & K_\bullet(C^*(\mathcal{O}\partial X)) \\ \cong \Big\downarrow \exp_* & \circlearrowright & \cong \Big\downarrow \exp_* \\ KX_\bullet(X) & \xrightarrow{\mu_X} & K_\bullet(C^*(X)) \end{array}$$

Theorem (Higson-Roe)

Coarse Baum-Connes conjecture for open cones over compact metrizable spaces (especially, $\mathcal{O}\partial X$) holds.

QED.

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Sketch of the proof of Main Theorem

We follow Higson-Roe's argument (for Gromov hyperbolic space)

STEP1 Show "log" is a coarse homotopy inverse of exp.

$$\log \colon \mathrm{Image}(\exp) \ni x \mapsto t^{\epsilon}[\gamma] \in \mathcal{O}\partial X$$

where $t := \overline{O, x}$ and $\gamma \in \mathcal{L}_{O}^{\infty}$ s.t. $\gamma(t) = x$.
Remark: exp is not necessarily coarsely surjective.

STEP2 Construct an appropriate map

$$r \colon X \to \mathrm{Image}(\exp)$$

and show this is coarsely homotopy equivalent map.

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Unlike Gromov hyperbolic space, Image(exp) *is not necessarily quasi-convex subset.*

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Coarse Equivalence

Let X, Y be metric spaces and $f\colon X \to Y$ be a map

- f is bornologous if $\exists \rho\colon [0,\infty) \to [0,\infty)$ s.t.

$$\forall p, q \in X,\ \overline{f(p), f(q)} < \rho(\overline{p, q}).$$

- f is proper if $B \subset Y$: bounded $\Rightarrow f^{-1}(B)$: bounded
- f is coarse if f is proper and bornologous.

Let $f, g\colon X \to Y$ maps.

- f and g are close if $\exists C > 0$, $\forall p \in X$, $\overline{f(p), g(p)} < C$.

X and Y are coarsely equivalent if $\exists f\colon X \to Y, \exists g\colon Y \to X$ s.t.

1. f and g are coarse maps,
2. $g \circ f$ is close to id_X,
3. $f \circ g$ is close to id_Y.

Coarse Equivalence

Let X, Y be metric spaces and $f\colon X \to Y$ be a map

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Coarsely Homotopy Equivalent

$f, g\colon X \to Y$: coarse maps

Definition

f and g are coarsely homotopic if
$\exists Z = \{(x,t) : 0 \leqslant t \leqslant T_x\} \subset X \times \mathbb{R}_{\geqslant 0}$, $\exists h\colon Z \to Y$: coarse map, s.t.

1. the map $X \ni x \mapsto T_x \in \mathbb{R}_{\geqslant 0}$ is bornologous,
2. $h(x, 0) = f(x)$, and
3. $h(x, T_x) = g(x)$.

X and Y are coarsely homotopy equivalent if
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